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Orthogonalized Fourier Polynomials for Signal Approximation and Transfer (Supplementary Materials)

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In this document, we report the full derivation for the iterative computation of the transfer matrix $\mathbf{O}$ which, due to lack of space, we did not include in the main manuscript.

1. An iterative formula for the transform O

We consider $\Pi : \mathcal{N} \to \mathcal{M}$ the point-to-point correspondence between two shapes $\mathcal{N}$ and $\mathcal{M}$. $T_F : L^2(\mathcal{M}) \to L^2(\mathcal{N})$ is the functional map associated to this correspondence defined via pull-back $T_F(f) = f \circ \pi$, $\forall f \in L^2(\mathcal{M})$. $\mathbf{C}$ is the matrix representation of T_F in a truncated pair of bases $\mathbf{\Phi}$ and $\mathbf{\Psi}$ for $L^2(\mathcal{M})$ and $L^2(\mathcal{N})$ respectively. For simplicity we consider both the finite bases with the same dimension K and the general case directly arise from our analysis. $\tilde{\mathbf{\Phi}}$ and $\tilde{\mathbf{\Psi}}$ are the matrices the columns of which are the eigenproducts of order N of the basis functions contained in $\mathbf{\Phi}$ and $\mathbf{\Psi}$. We represent each of these eigenproducts as $\tilde{\varphi}_h$ and $\tilde{\psi}_\ell$ $\tilde{\mathbf{C}}$ is the funxtional maps extended to the eigenproducts following the formula proposed in [NMR*18].

As we describe in the main document, our bases are obtained by applying the Gram-Schmidt algorithm to $\tilde{\mathbf{\Phi}}$ and $\tilde{\mathbf{\Psi}}$ obtaining the two couples $\tilde{\mathbf{\Phi}} = \mathbf{Q}^{\Phi}\mathbf{R}^{\Phi}$ and $\tilde{\mathbf{\Psi}} = \mathbf{Q}^{\Psi}\mathbf{R}^{\Psi}$. we are therefore interested in estimating a transform $\mathcal{O}$ such that $T_F(\mathbf{Q}^{\Phi}) = \mathbf{Q}^{\Psi}\mathcal{O}$.

First of all due to the role of $\mathbf{C}$ we can write $\mathbf{\Psi}\mathbf{C} = T_F(\mathbf{\Phi})$ and $\tilde{\mathbf{\Psi}}\tilde{\mathbf{C}} = T_F(\tilde{\mathbf{\Phi}})$. These equality could be only approximation depending on the quality of the maps $\mathbf{C}$ and $\tilde{\mathbf{C}}$, and in the alignment of the bases involved. For this reason we can already set the first K columns of $\mathbf{O} = \begin{bmatrix}\mathbf{C}\\ \mathbf{0}\end{bmatrix}$, where $\mathbf{0}$ is a matrix of zeros with K columns and the number of rows equal to the number of functions in $\mathbf{Q}^{\Psi}$ minus K. This allows us to consider the first K columns of $\mathbf{O}$ already computed and only estimated the remaining ones. For this reason we look for an iterative construction of $\mathbf{O}$ that estimate at each iteration a new column of $\mathbf{O}$ from left to right. This procedure is related with the iterative construction of the bases $\mathbf{Q}^{\Phi}$ and $\mathbf{Q}^{\Psi}$.

Let us start writing the explicit formula for the column ζ_i of $\mathbf{Q}^{\Phi}$:

$$\zeta_i = \frac{1}{\mathbf{R}^{\Phi}_{i,i}}(\tilde{\varphi}_i - \sum_{h=1}^{i-1} \mathbf{R}^{\Phi}_{i,h}\zeta_h), \tag{1}$$

and in the same way:

$$\xi_j = \frac{1}{\mathbf{R}^{\Psi}_{j,j}}(\tilde{\psi}_j - \sum_{\ell=1}^{j-1} \mathbf{R}^{\Psi}_{i,\ell}\xi_\ell). \tag{2}$$

Now we want to compute the image via T_F of each function ζ_i (column) of $\mathbf{Q}^{\Phi}$, for $i > K$, thanks to the linearity of T_F we have:

$$T_F(\zeta_i) = T_F\Big(\frac{1}{\mathbf{R}^{\Phi}_{i,i}}(\tilde{\varphi}_i - \sum_{h=1}^{i-1} \mathbf{R}^{\Phi}_{i,h}\zeta_h)\Big) = \tag{3}$$

$$\frac{1}{\mathbf{R}^{\Phi}_{i,i}}(T_F(\tilde{\varphi}_i) - \sum_{h=1}^{i-1} \mathbf{R}^{\Phi}_{i,h}T_F(\zeta_h)). \tag{4}$$

Now we can consider that:

$$T_F(\tilde{\varphi}_i) = \tilde{\mathbf{\Psi}}\tilde{\mathbf{C}}_{:,i} \tag{5}$$

$$T_F(\zeta_h) = \sum_j \mathbf{O}_{j,h}\xi_j, \tag{6}$$

where j goes from 1 to the number of functions in $\mathbf{Q}^{\Psi}$ while $\tilde{\mathbf{C}}_{:,i}$ is the i-th column of $\tilde{\mathbf{C}}$. Equation 6 can be written only because we already know all the elements of $\mathbf{O}$, $\forall j$ and $\forall h \leq i-1$. Then we can substitute the equivalences 5 and 6 in 4:

$$T_F(\zeta_i) = \frac{1}{\mathbf{R}^{\Phi}_{i,i}}\Big(\tilde{\mathbf{\Psi}}\tilde{\mathbf{C}}_{:,i} - \sum_{h=1}^{i-1} \mathbf{R}^{\Phi}_{i,h}(\sum_j \mathbf{O}_{j,h}\xi_j)\Big). \tag{7}$$

Now we know that $\tilde{\mathbf{\Psi}} = \mathbf{Q}^{\Psi}\mathbf{R}^{\Psi}$ so we can write:

$$T_F(\zeta_i) = \frac{1}{\mathbf{R}^{\Phi}_{i,i}}\Big(\mathbf{Q}^{\Psi}\mathbf{R}^{\Psi}\tilde{\mathbf{C}}_{:,i} - \sum_{h=1}^{i-1} \mathbf{R}^{\Phi}_{i,h}(\sum_j \mathbf{O}_{j,h}\xi_j)\Big) = \tag{8}$$

$$\frac{1}{\mathbf{R}^{\Phi}_{i,i}}\Big(\sum_j \sum_{l=1} \mathbf{R}^{\Psi}_{j,l}\tilde{\mathbf{C}}_{l,i}\xi_j - \sum_{h=1}^{i-1} \mathbf{R}^{\Phi}_{i,h}(\sum_j \mathbf{O}_{j,h}\xi_j)\Big), \tag{9}$$

where the last equation comes from the definition of the matrix product. Now we can collect and reorder the element in the last equation with respect to the sum over j:

$$T_F(\zeta_i) = \frac{1}{\mathbf{R}^{\Phi}_{i,i}} \Big(\sum_j \underbrace{(\sum_{l=1} \mathbf{R}^{\Psi}_{j,l} \tilde{\mathbf{C}}_{l,i} - \sum_{h=1}^{i-1} \mathbf{R}^{\Phi}_{i,h} \mathbf{O}_{j,h})}_{\mathbf{O}_{j,i}} \xi_j \Big), \tag{10}$$

where $\mathbf{O}_{j,i}$ only depends on $\mathbf{R}^{\Phi}, \tilde{\mathbf{C}}, \mathbf{R}^{\Psi}$ and form the first $i-1$ columns of $\mathbf{O}$. This result proves that $\mathbf{O}$ can be iteratively computed on its columns as a function of known variables. Moreover, being $\tilde{\mathbf{C}}$ a function of $\mathbf{C}$, this result clarify once again that it is possible to fully recover $\mathbf{O}$ from $\mathbf{C}$ justifying the proposed procedure.

References

[NMR*18] NOGNENG D., MELZI S., RODOLÀ E., CASTELLANI U., BRONSTEIN M., OVSJANIKOV M.: Improved functional mappings via product preservation. *Computer Graphics Forum 37*, 2 (2018), 179–190. 1